Tutorial 1: A Do-It-Yourself Guide to Lex-Solving

In this tutorial we assume that $K = \mathbb{Q}$ and let $P = K[x_1, \dots, x_n]$. (With some modifications and additions, you can make the following work over a finite field but for simplicity we shall stick with the rational case.) Let $f_1, \dots, f_s \in P \setminus \{0\}$ be polynomials defining a system of equations

$$\begin{aligned} f_1(x_1, \dots, x_n) &= 0 \\ &\vdots \\ f_s(x_1, \dots, x_n) &= 0 \end{aligned}$$

and let $I = \langle f_1, \dots, f_s \rangle \subseteq P$.

a) Write a CoCoA function `IsVarPower(...)` which takes an integer $i \in \{1, \dots, n\}$ and a term $t \in \mathbb{T}^n$, checks whether t is a power of x_i and returns the corresponding Boolean value.

b) Implement a CoCoA function `IsZeroDim(...)` which uses the last condition of the finiteness criterion to check whether I is zero-dimensional.

   Hints: The last condition is "for every $i \in \{1, \dots, n\}$ there exists a number $\alpha_i > 0$ such that we have $x_i^{\alpha_i} \in \mathrm{LT}_\sigma(I)$". Use only one call to a CoCoA function which involves a Gröbner basis computation, namely `LT(I)`.

c) Apply your function `IsZeroDim(...)` to check which of the following ideals are zero-dimensional.

   1. $I_1 = \langle x_1x_3 - x_2^2,\ x_1x_4 - x_2x_3,\ x_2x_4 - x_3^2,\ x_2^2x_3 - x_1x_3^2 \rangle \subseteq \mathbb{Q}[x_1, \dots, x_4]$
   2. $I_2 = \langle x_1^3 - x_2x_3^2,\ x_1^2x_2x_3 - x_2^4,\ x_1^2 + x_2^3 + x_3^2 \rangle \subseteq \mathbb{Q}[x_1, x_2, x_3]$
   3. $I_3 = \langle x_1x_2 - x_3^2,\ x_2^2 - x_3x_4,\ x_1x_3 - x_4^3,\ x_2x_4 - x_3^2 \rangle \subseteq \mathbb{Q}[x_1, \dots, x_4]$
   4. $I_4 = \langle x_1^2 - x_1x_2,\ x_2^2 - x_2x_3,\ x_3^2 - x_3x_1,\ x_1x_2 + x_2x_3 + x_1x_3 \rangle \subseteq \mathbb{Q}[x_1, x_2, x_3]$

d) Show that the squarefree part of a polynomial $g \in P \setminus \{0\}$ can be computed by the formula $\mathrm{sqfree}(f) = f / \gcd(f, f')$. Then write a CoCoA function `SqFree(...)` which does this.

e) Write a CoCoA function `ZeroDimRadical(...)` which uses the following result to compute $\sqrt{I}$: *For* $i = 1, \dots, n$, *let* $\langle g_i \rangle = I \cap K[x_i]$. *Then we have* $\sqrt{I} = I + \langle \mathrm{sqfree}(g_1), \dots, \mathrm{sqfree}(g_n) \rangle$.

f) Using your function `ZeroDimRadical(...)`, compute the radicals of the following ideals. Then use your results to analyze their zeros and the multiplicities of these zeros.

   1. $J_1 = \langle x^3,\ x^2y + x,\ y^2 \rangle \subseteq \mathbb{Q}[x, y]$
   2. $J_2 = \langle x^2 + 2xy + y^2,\ xz + yz,\ xy^2 + y^3 + xy + y^2,\ y^4 + 2y^3 + y^2,\ y^2z + yz,\ z^3 - xy \rangle \subseteq \mathbb{Q}[x, y, z]$
   3. $J_3 = \langle x^2 + y + z - 1,\ x + y^2 + z - 1,\ x + y + z^2 - 1 \rangle \subseteq \mathbb{Q}[x, y, z]$

4. $J_4 = \langle x^3 - 3x^2y + 3xy^2 - y^3 - z^2,\ -x^3 + 3x^2z - 3xz^2 + z^3 - y^2,\ y^3 - 3y^2z + 3yz^2 - z^3 - x^2\rangle \subseteq \mathbb{Q}[x,y,z]$

g) Write a CoCoA function `IsNormalPos(...)` which checks whether a zero-dimensional radical ideal I is in normal x_i-position for some $i \in \{1,\dots,n\}$ and returns 0 (for "false") or the corresponding i.

h) Apply your function `IsNormalPos(...)` to check whether the following zero-dimensional ideals are in normal x_i-position for some $i \in \{1,\dots,n\}$.

1. $J_5 = \langle x^2 + y^2 - 1,\ 4xy - 2x - 2y + 1\rangle \subseteq \mathbb{Q}[x,y]$
2. $J_6 = \langle x^2 - y,\ x^2 - 3x + 2\rangle \subseteq \mathbb{Q}[x,y]$
3. $J_7 = \langle yz + z,\ y^2 + y,\ x + y + z,\ z^2 - z\rangle \subseteq \mathbb{Q}[x,y,z]$

i) Suppose that I is not in normal x_i-position for any $i \in \{1,\dots,n\}$. Write a CoCoA function `NormalPosTrafo(...)` which computes an index $i \in \{1,\dots,n\}$ and a tuple $(c_1,\dots,c_{n-1}) \in K^{n-1}$ such that the linear change of coordinates which is defined by $x_j \mapsto x_j$ for $j \neq i$ and by $x_i \mapsto x_i - c_1x_1 - \cdots - c_{i-1}x_{i-1} - c_ix_{i+1} - \cdots - c_{n-1}x_n$ transforms I into an ideal in normal x_i-position.

j) Use your function `NormalPosTrafo(...)` to find transformations which bring the following ideals into normal x_i-position for some $i \in \{1,\dots,n\}$.

1. $J_8 = \langle x^2 + y^2 + 3,\ x^2 - y^2 + 1\rangle \subseteq \mathbb{Q}[x,y]$
2. $J_9 = \langle xy,\ x^2 + x,\ y^3 + 2y^2 + y\rangle \subseteq \mathbb{Q}[x,y]$
3. $J_{10} = \langle x^3 - x,\ y^2 - y,\ z^2 - z\rangle \subseteq \mathbb{Q}[x,y,z]$

k) Now combine your previous functions and Corollary 3.7.26 (Solving Systems Effectively) to implement your own `LexSolver(...)`. The function should return a tuple of polynomials $(g_0,\dots,g_n) \in P^{n+1}$ such that $\mathcal{Z}(I)$ consists of the points $(g_1(a_i),\dots,g_n(a_i)) \in \overline{\mathbb{Q}}^n$ where the a_i are the roots of g_0.

l) Apply your `LexSolver(...)` to the following ideals. When necessary, use the built-in CoCoA function `RealRoots(...)` to find good approximations for the real roots of the system.

1. The zero-dimensional ideals in (c).
2. The ideals $J_1 - J_{10}$.
3. $F_1 = \langle y^2 - \frac{2}{375}xz + \frac{22}{75}yz + \frac{29}{1500}z^2 + \frac{2}{75}x - \frac{7}{15}y - \frac{7}{150}z,\ xy + \frac{31}{150}xz - \frac{1}{5}yz - \frac{1}{50}z^2 - \frac{8}{15}x + y + \frac{1}{10}z,\ x^2 - \frac{7}{15}xz + 4yz + \frac{2}{5}z^2 + \frac{7}{3}x - 20y - 2z,\ z^3 + \frac{6}{5}xz + 24yz - \frac{38}{5}z^2 - 6x - 120y + 13z,\ yz^2 - \frac{3}{25}xz - \frac{47}{5}yz + \frac{3}{50}z^2 + \frac{3}{5}x + 22y - \frac{3}{10}z,\ xz^2 - 7xz + 10x\rangle \subseteq \mathbb{Q}[x,y,z]$
4. $F_2 = \langle x^2 + y^2 + z^2 - 9,\ 3x^2 - y^2z,\ x^2z - 2y^2 + 2\rangle \subseteq \mathbb{Q}[x,y,z]$
5. $F_3 = \langle 24xy - x^2 - y^2 - x^2y^2 - 13,\ 24xz - x^2 - z^2 - x^2z^2 - 13,\ 24yz - y^2 - z^2 - y^2z^2 - 13\rangle \subseteq \mathbb{Q}[x,y,z]$